ATTACHMENT OF EDDY FLOWS OF AN IDEAL FLUID

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It is known that the solution of the equation

$$\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 = F(\psi), \tag{1}$$

where the vorticity F is an arbitrary function of ψ, can be considered as an example of steady-state flow of an ideal fluid. If we suppose that the motion of an ideal incompressible fluid can be thought of as the threshold motion of a viscous fluid, the function $F(\psi)$ in Eq. (1) can be replaced by a constant [1].

Let us consider the following simulation problem with cohesively selected piecewise-constant vorticity. In a bounded region D with boundary Γ it is necessary to find a continuously differentiable solution of the equation

$$\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 = \begin{cases} \omega, & \text{if} \quad \psi < 0 \\ -\omega_1, & \text{if} \quad \psi > 0 \end{cases} \tag{2}$$

(ω and ω_1 are nonnegative constants) under the boundary condition

$$\psi|_\Gamma = \varphi(s). \tag{3}$$

If we set $\omega_1 = 0$ in Eq. (2), we obtain an equation that describes the motion of an ideal fluid according to a previous scheme [2]. This type of flow for the case of a bounded region [3] and for the case of an unbounded region has been studied earlier [4-7].

The problem (2), (3) has the trivial solution

$$\psi = \varphi_0 + \frac{\omega_1}{2\pi}\iint\limits_D G\,d\xi\,d\tau,$$

where φ_0 is a harmonic function satisfying condition (3) and G is Green's function of the region D of the Dirichlet problem for the Laplacian. In [3] it was proved that a nontrivial solution for the case $\omega_1 = 0$ exists under particular conditions. We will derive a condition under which a nontrivial solution of the problem (2), (3) exists. A simpler bound than in [3] will be obtained from this condition for $\omega_1 = 0$.

Suppose $\varphi(s) \le C$ and let B_1 be the circle of greatest radius, such that $B_1 \subseteq D$ (without loss of generality we may assume that its center coincides with the coordinate origin), and let B_2 be the circle of least radius with center at the origin, such that $B_2 \supseteq D$. The radius of B_1 is R_1 and that of B_2, R_2. We have the following assertion: When

$$\omega - \frac{\omega_1 R_2^2}{R_1^2}\,e \geqslant \frac{4Ce}{R_1^2} \tag{4}$$

the problem (2), (3) has a nontrivial solution. Let us prove this assertion. If the circle B_1 is taken as the region D and if we set $\omega_1 = 0$ in Eq. (2), and let $\varphi(s) = C + \omega_1 R_2^2/4$ in Eq. (3), whenever (4) holds, the problem has two nontrivial solutions (found explicitly). That is, in particular, there exists a circle $B_a < B_1$ of radius a such that the corresponding solution is negative.

Let us consider the auxiliary problem

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$$\frac{\partial^2\psi_n}{\partial x^2}+\frac{\partial^2\psi_n}{\partial y^2}=\begin{cases}\omega, & \text{if } x, y\in B_a\\ \frac{\omega}{2}(1-\operatorname{th}\psi_n n)-\frac{\omega_1}{2}(1+\operatorname{th}\psi_n n), & \text{if } x, y\in D\backslash\overline{B}_a;\end{cases} \tag{5}$$

$$\psi_n|_\Gamma=\varphi(s). \tag{6}$$

The solution will be found in the class of functions continuously differentiable in D. The problem (5), (6) is equivalent to the integral equation

$$\psi_n=\varphi_0-\frac{\omega}{2\pi}\iint\limits_{B_a} G d\xi d\tau+\frac{1}{4\pi}\iint\limits_{D\backslash B_a}[\omega_1(1+\operatorname{th}\psi_n n)-\omega(1-\operatorname{th}\psi_n n)]\,G d\xi d\tau. \tag{7}$$

The Schauder theorem can be used to establish the existence of a solution of Eq. (7) for any n and $x, y \in D\backslash\overline{B}_a$. We substitute this solution in the right side of Eq. (7), thus defining the function ψ_n over all of D. The resulting function is the solution of the problem (5), (6). It follows from the properties of a potential-type integral that it has first derivatives in every fixed closed region $\overline{B}\subset D$; these derivatives satisfy the Hölder condition, while the constant and exponent are independent of n.

We use the Arzelà theorem to establish that the sequence ψ_n is compact in the space of continuously differentiable functions. Suppose the subsequence ψ_{n_k} converges to a continuously differentiable function ψ^*. We will prove that ψ^* is a nontrivial solution of the problem (2), (3).

Suppose that $y_0 \in D\backslash\overline{B}_a$ and $\psi^*(x_0, y_0) > 0$ at some point x_0. It will then be greater than zero also in some circular neighborhood. We now consider Eq. (5) in this neighborhood and take its limit as $n_k \to \infty$, obtaining $\partial^2\psi^*/\partial x^2+\partial^2\psi^*/\partial y^2=-\omega_1$. It can be analogously proved that $\partial^2\psi^*/\partial x^2+\partial^2\psi^*/\partial y^2=\omega$ at points at which $\psi^*<0$. Further, when $x,y\in B_a$, $\partial^2\psi^*/\partial x^2+\partial^2\psi^*/\partial y^2=\omega$. Let us prove that when x, y $\in B_a$, $\psi^*<0$. It follows from the properties of Green's function that

$$\begin{gathered}\frac{1}{2\pi}\iint\limits_{D} G d\xi d\tau\leqslant\frac{1}{2\pi}\iint\limits_{B_2} G_{B_2} d\xi d\tau\leqslant\frac{R_2^2}{4};\\ \iint\limits_{B_a} G d\xi d\tau\geqslant\iint\limits_{B_a} G_{B_1} d\xi d\tau\,(x, y\in B_1),\end{gathered} \tag{8}$$

where G_{B_1} and G_{B_2} are Green's functions for the regions B_1 and B_2, respectively. We find from Eqs. (7) and (8) that

$$\psi_n<V=C+\frac{\omega_1R_2^2}{4}-\frac{\omega}{2\pi}\iint\limits_{B_a} G_{B_1} d\xi d\tau,$$

It follows from the definition of B_a that the function V is negative in B_a. Then ψ_n, that is, ψ^*, are both negative in B_a. The fact that ψ^* satisfies the equation as we pass through the boundary of B_a follows from its smoothness.

We set $\omega_1=0$ in Eq. (4), obtaining the condition $\omega \geq 4Ce/R_1^2$ under which there exists a nontrivial solution of the problem describing flow in the M. A. Lavrent'ev scheme for the case of a bounded region.

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